NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1

2020 Year 12 Course Assessment Task 4 (Trial Examination)
Thursday August 27, 2020

General instructions

- Working time – 2 hours.
  (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided (on page 11)

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED**: .....

Class (please ✔)

◯ 12MXX.1 – Ms Ham ◯ 12MAX.3 – Mr Sekaran ◯ 12MAX.5 – Mr Siu
◯ 12MXX.2 – Mr Lam ◯ 12MAX.4 – Ms Bhamra

Marker's use only.

QUESTION	1-10	11	12	13	14	Total	%
MARKS	/10	/17	/15	/14	/14	/70	

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

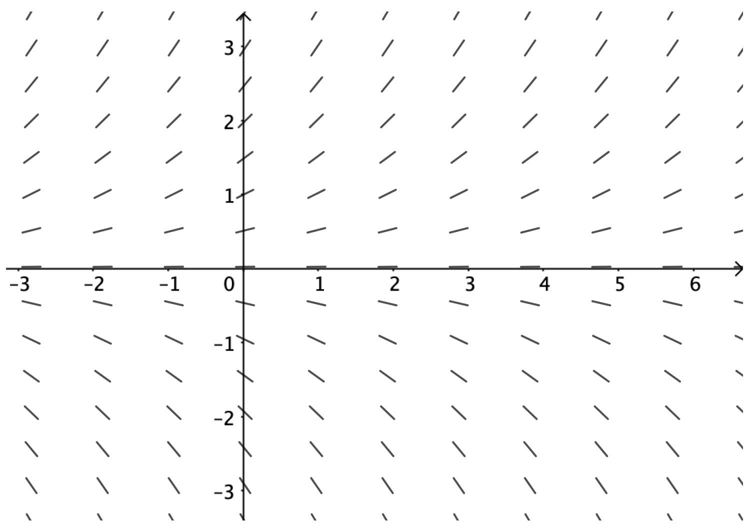
Mark your answers on the answer grid provided (labelled as page 11).

Questions **Marks**

1. What is the exact value of $\sec\left(-\dfrac{5\pi}{3}\right)$? **1**

(A) $\dfrac{1}{2}$ (B) $-\dfrac{1}{2}$ (C) -2 (D) 2

2. Consider the slope field shown below. **1**



What is a possible differential equation for the slope field?

(A) $\dfrac{dy}{dx} = -\dfrac{x}{2}$ (B) $\dfrac{dy}{dx} = \dfrac{x}{2}$ (C) $\dfrac{dy}{dx} = -\dfrac{y}{2}$ (D) $\dfrac{dy}{dx} = \dfrac{y}{2}$

3. What is the smallest positive x value for which $\sin x + \cos x$ is at its maximum? **1**

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{\pi}{4}$ (C) π (D) $\dfrac{\pi}{3}$

4. Which one of the following transformations is **not** required to obtain $g(x) = -1 + \frac{1}{2} f(3 - 2x)$ from $f(x)$? **1**

(A) A reflection in the y axis

(B) A reflection in the x axis

(C) A translation parallel to the x axis

(D) A compression in the y direction

5. Which one of the following is the correct description of the asymptote(s) of **1**

$$y = \frac{1}{x^2 + 7x - 8}$$

(A) exactly one straight line asymptote.

(B) exactly two straight line asymptotes.

(C) exactly three straight line asymptotes.

(D) $x = -1$ and $x = 8$ as its vertical asymptotes.

6. Consider the polynomial $P(x) = x^3 + x^2 + cx - 10$. It is known that two of its zeros are equal in magnitude but opposite in sign. **1**

What is the value of c?

(A) $\sqrt{10}$ (B) 10 (C) -10 (D) $-\sqrt{10}$

7. Which expression is equal to $\int \cos 3x \, \cos 2x \, dx$? **1**

(A) $\frac{1}{10}\sin 5x + \frac{1}{2}\sin x + c$

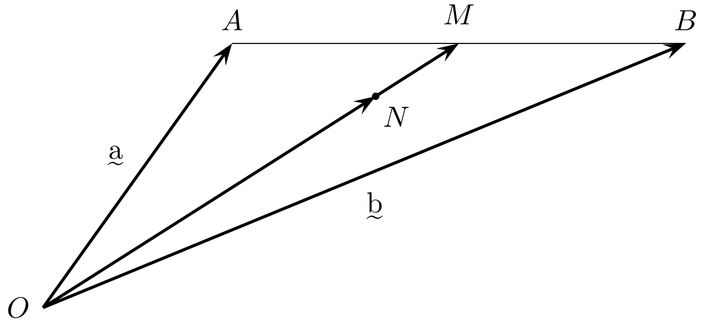
(B) $\frac{1}{2}(\cos 5x + \cos x) + c$

(C) $\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + c$

(D) $\frac{1}{2}(\sin 5x + \sin x) + c$

8. In the following diagram, M is the midpoint of line segment AB, and $\overrightarrow{ON} = \frac{4}{5}\overrightarrow{OM}$. **1**

It is given that $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



Which of the following expressions correctly relates $\overrightarrow{NM}$ with $\underset{\sim}{a}$ and $\underset{\sim}{b}$?

(A) $\overrightarrow{NM} = \frac{4}{5}(\underset{\sim}{a} - \underset{\sim}{b})$

(B) $\overrightarrow{NM} = \frac{2}{5}(\underset{\sim}{a} + \underset{\sim}{b})$

(C) $\overrightarrow{NM} = \frac{1}{10}(\underset{\sim}{a} + \underset{\sim}{b})$

(D) $\overrightarrow{NM} = \frac{1}{10}(\underset{\sim}{a} - \underset{\sim}{b})$

9. What is the limiting sum of the following geometric series? **1**

$$1 - \cos(2x) + \cos^2(2x) - \cos^3(2x) + \ldots$$

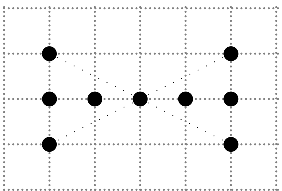
(A) $\frac{1}{2}\sec^2 x$

(B) $\frac{1}{2}\operatorname{cosec}^2 x$

(C) $\frac{1}{2}\cos^2 x$

(D) $\frac{1}{2}\sin^2 x$

10. The diagram below shows a shape made by 9 points. **1**



How many combinations of 3 points are collinear?

(A) 10 (B) 14 (C) 12 (D) 8

Examination continues overleaf...

Section II

60 marks
Attempt Questions 11 to 14
Allow approximately 1 hours and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (17 Marks) Commence a NEW booklet. **Marks**

(a) Given the function $f(x) = \sqrt{2x+1}$ and that $f^{-1}(x)$ is the inverse function of $f(x)$, find $f^{-1}(5)$. **2**

(b) Solve for x: **3**

$$\frac{x}{x+1} > 2$$

(c) Use the method of addition of ordinates to sketch the graph of **3**

$$f(x) = e^{-x} - \frac{x}{e}$$

given $e \approx 2.7$. It is also given that the x-intercept is $x = 1$. Show all important features.

(d) i. Use t-formulae to show that **1**

$$\frac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2}.$$

ii. Hence, or otherwise, find the exact value of $\tan 15°$. **1**

(e) Find the exact value of $\tan\left(2\sin^{-1}\frac{2}{3}\right)$ **2**

(f) At a soccer club a team of 13 players is to be chosen from a pool of 25 players consisting of 20 female players and 5 male players. **2**

What is the probability that the team will consist of only female players?

(g) Find the term independent of x in the expansion of $\left(3x - \frac{1}{2x^2}\right)^9$ **3**

Examination continues overleaf...

Question 12 (15 Marks) Commence a NEW booklet. **Marks**

(a) If $x = \alpha$ is a double root of a polynomial $P(x)$, show that $x = \alpha$ is also a root of $P'(x)$. **2**

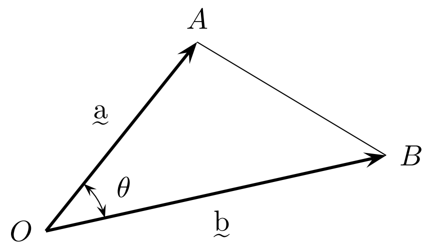
Hint: Let $P(x) = (x - \alpha)^2 Q(x)$

(b) A force, described by the vector $\underset{\sim}{F} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, moves a particle along the line ℓ from $A(-1, -1)$ to $B(2, -3)$.

i. Find the unit vector that is in the direction of $\overrightarrow{AB}$. **2**

ii. Find the component of $\underset{\sim}{F}$ in the direction of line ℓ. **2**

(c) The diagram below shows ΔOAB, where $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. **3**



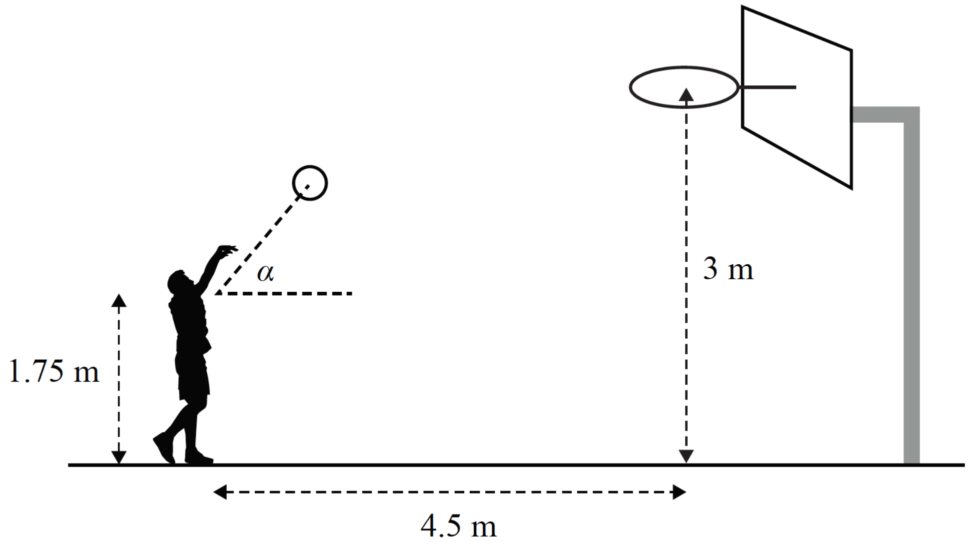
Given that θ is the angle between vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, show that the area of the triangle is

$$A = \frac{1}{2}\sqrt{\left(\underset{\sim}{a} \cdot \underset{\sim}{a}\right)\left(\underset{\sim}{b} \cdot \underset{\sim}{b}\right) - \left(\underset{\sim}{a} \cdot \underset{\sim}{b}\right)^2}$$

Examination continues overleaf…

Question 12 continued from previous page...

(d) A basketball player aims to throw a basketball through a ring, the centre of which is at a horizontal distance of 4.5 m from the point of release of the ball and 3 m above floor level. The ball is released at a height of 1.75 m above floor level, at an angle of projection α to the horizontal and at a speed of V ms^{-1}. Air resistance is assumed to be negligible.



The position vector of the centre of the ball at any time t seconds, for $t \geq 0$, relative to the point of release is given by

$$\underset{\sim}{r} = \begin{pmatrix} Vt\cos\alpha \\ Vt\sin\alpha - 5t^2 \end{pmatrix}$$

Displacement components are measured in metres. For the player's first shot at goal, $V = 7\,\text{ms}^{-1}$ and $\alpha = 45^\circ$.

i. Find the time, in seconds, taken for the ball to reach its maximum height. Give your answer in the form $\dfrac{a\sqrt{b}}{c}$, where a, b and c are positive integers. **2**

ii. Find the maximum height, in metres, above floor level, reached by the centre of the ball. **2**

Give your answer correct to two decimal places.

iii. Find the distance between the centre of the ball and the centre of the ring when the ball reaches its maximum height. **2**

Give your answer in metres, correct to two decimal places.

Examination continues overleaf...

Question 13 (14 Marks) Commence a NEW booklet. **Marks**

(a) Consider the differential equation $\dfrac{dy}{dx} - \dfrac{2y}{x} = 0$. Find the equation of the solution curve if it passes through $(1, -1)$. **3**

(b) Prove by mathematical induction that $3^{3n} + 2^{n+2}$ is a multiple of 5 for all positive integers n. **3**

(c) It is given that the rate of decrease of temperature of a body that is hotter than its surrounding air is proportional to the temperature difference.
i.e.

$$\frac{dT}{dt} = -k(T - A)$$

where A is the air temperature, and T is the temperature of the body after t minutes

i. Show that, if the initial temperature is I, then the following function satisfies the above differential equation. **1**

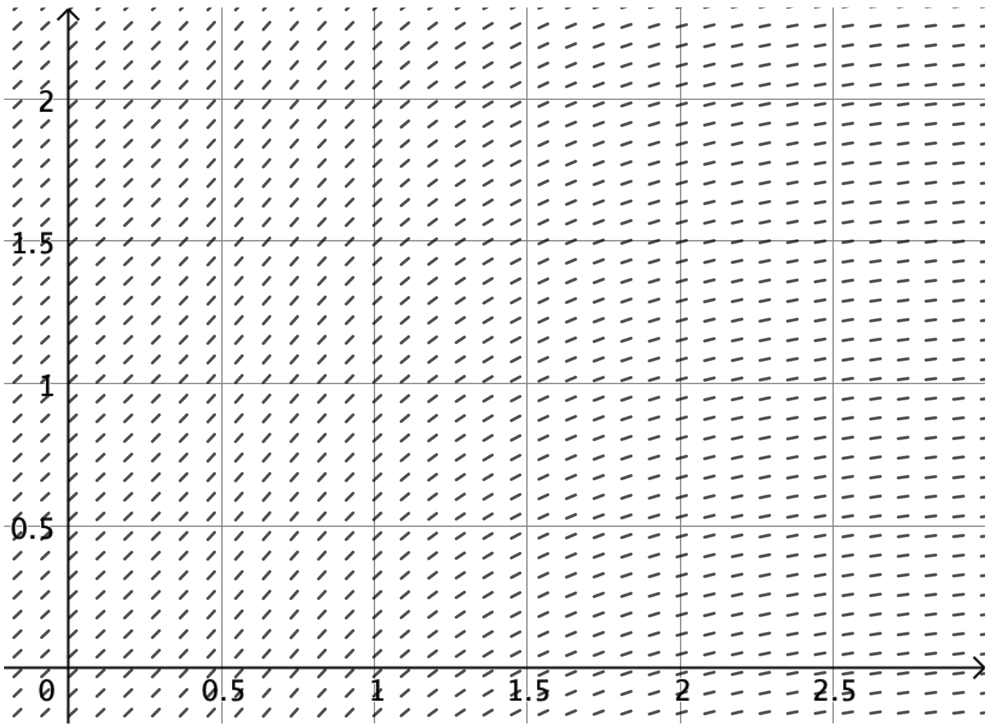
$$T = A + (I - A)e^{-kt}$$

ii. A block of iron, initially at a temperature of $1500°C$, is allowed to cool in the open air, where the temperature is $20°C$. If it cools to $1200°C$ in five minutes, find the temperature of the block after one hour, correct to 3 significant figures. **2**

Examination continues overleaf...

Question 13 continued from previous page...

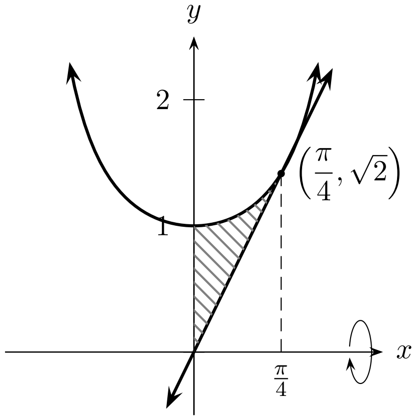
(d) The slope field of $\dfrac{dy}{dx} = \dfrac{1+x^2}{1+x^4}$ is shown below.



i. It is given that the solution curve passes through $(0.5, 0.5)$. On the grid provided on page 12, sketch the solution curve on the slope field. **1**

ii. Find the approximate value of y when $x = 2$. Give your answer correct to one decimal place. **1**

(e) The diagram below shows parts of the graph of both $y = \sec x$ and $y = \dfrac{4\sqrt{2}}{\pi}x$. **3**
It is given that the two functions intersect at $\left(\dfrac{\pi}{4}, \sqrt{2}\right)$ for the first time in the first quadrant.



The area bounded by the curve $y = \sec x, y = \dfrac{4\sqrt{2}}{\pi}x$ and the y axis is rotated about the x axis. Find the volume of the solid of revolution.

Examination continues overleaf...

Question 14 (14 Marks) Commence a NEW booklet. **Marks**

(a) Use the substitution $u = 1 + \sqrt{x}$ to evaluate **4**

$$\int_1^9 \frac{\sqrt{x}+2}{\sqrt{x}\sqrt{1+\sqrt{x}}}\,dx$$

, expressing the answer in the form $\sqrt{n}$ where n is a positive integer.

(b) The volume of water in a tank is given by $V = \frac{5\pi}{6} - \cos^{-1}\left(h - \frac{\sqrt{3}}{2}\right)$, where V is measured in m^3 and h is depth of the water in the tank in metres.

i. Find the domain and range of $V = \frac{5\pi}{6} - \cos^{-1}\left(h - \frac{\sqrt{3}}{2}\right)$. **2**

ii. Sketch the graph of $V = \frac{5\pi}{6} - \cos^{-1}\left(h - \frac{\sqrt{3}}{2}\right)$, showing all important features. **3**

From time $t = 0$, where t is measured in minutes, water is pumped into the empty tank at a constant rate of 50 litres per minute.
(*Note*: $1\,\text{m}^3 = 1\,000\,\text{L}$)

iii. Find the expression of V in terms of t. **1**

iv. Find the exact time in minutes required to reach the maximum volume of water. **1**

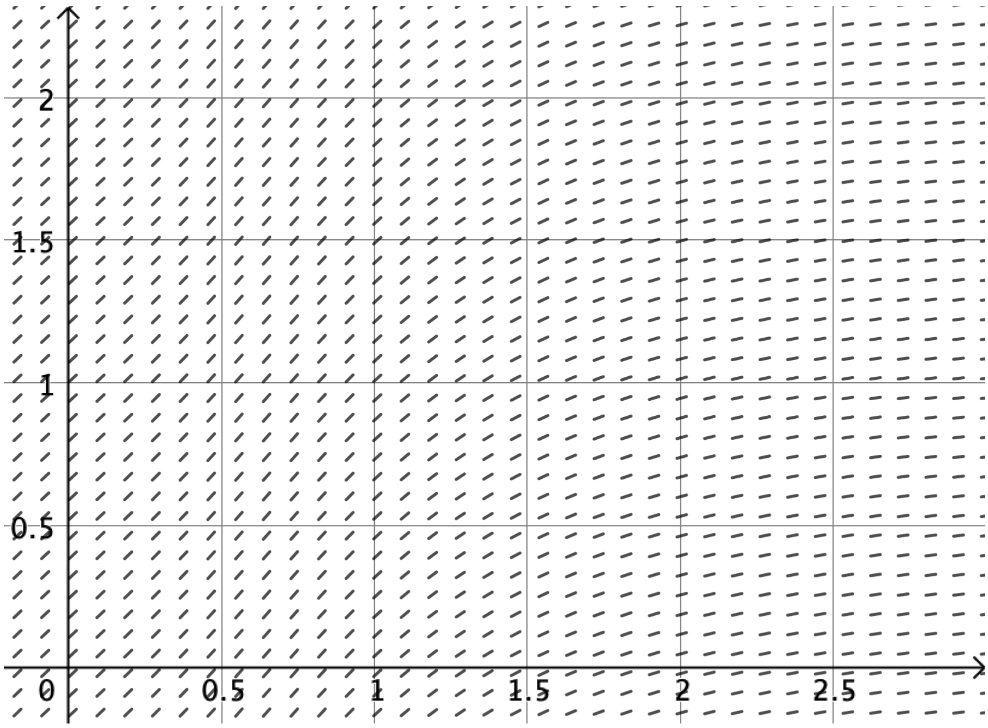
v. Find the rate of increase of the depth when the water in the tank is $\sqrt{3}$ metres deep. **3**

End of paper.

Slope field

Question 13

(d) i. It is given that the solution curve passes through $(0.5, 0.5)$. Sketch the solution curve on the slope field below. See Question 13((d))i on page 9. **1**



Sample Band 6 Responses

Section I

1. (D) **2.** (D) **3.** (B) **4.** (B) **5.** (C)
6. (C) **7.** (A) **8.** (C) **9.** (A) **10.** (B)

Section II

Question 11

(a) (2 marks)

- ✓ [1] for an equation involving $f(x)$
- ✓ [1] for final answer

$$
\begin{aligned}
5 &= \sqrt{2x+1} \\
&\text{Squaring both sides,} \\
25 &= 2x+1 \\
2x &= 24 \\
x &= 12 \\
\therefore f^{-1}(5) &= 12
\end{aligned}
$$

(b) (3 marks)

- ✓ [1] for multiplying both sides by $(x+1)^2$
- ✓ [1] for the correct quadratic inequality
- ✓ [1] for final answer

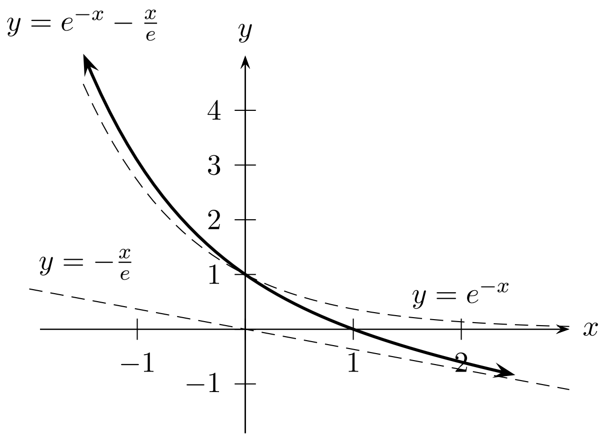
$$\frac{x}{x+1} > 2$$

Multiply both sides by $(x+1)^2$

$$
\begin{aligned}
x(x+1) &> 2(x+1)^2 \\
2(x+1)^2 - x(x+1) &< 0 \\
(x+1)\left(2(x+1) - x\right) &< 0 \\
(x+1)(x+2) &< 0 \\
\therefore -2 < x &< -1
\end{aligned}
$$

(c) (3 marks)

- ✓ [1] for shape
- ✓ [1] for y-intercept
- ✓ [1] for approaching $y = -\dfrac{x}{e}$ as $x \to \infty$



(d) i. (1 mark)

$$
\begin{aligned}
\text{LHS} &= \frac{1-\cos\theta}{\sin\theta} \\
&= \frac{1 - \left(\frac{1-t^2}{1+t^2}\right)}{\frac{2t}{1+t^2}} \\
&= \frac{1+t^2-1+t^2}{2t} \\
&= \frac{2t^2}{2t} \\
&= t \\
&= \text{RHS}
\end{aligned}
$$

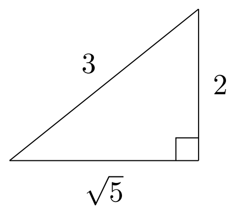
ii. (1 mark)

$$
\begin{aligned}
&\text{Let } \theta = 30^\circ \\
\tan 15^\circ &= \frac{1-\cos 30^\circ}{\sin 30^\circ} \\
&= \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} \\
&= 2 - \sqrt{3}
\end{aligned}
$$

(e) (2 marks)

- ✓ [1] for $\tan A$
- ✓ [1] for final answer

$$
\begin{aligned}
\text{Let } A &= \sin^{-1}\frac{2}{3} \\
\sin A &= \frac{2}{3}
\end{aligned}
$$



$$\begin{aligned}\tan 2A &= \frac{2\tan A}{1-\tan^2 A}\\ &= \frac{2\times\frac{2}{\sqrt{5}}}{1-\frac{4}{5}}\\ &= \frac{4}{\sqrt{5}}\times 5\\ &= 4\sqrt{5}\end{aligned}$$

(f) (2 marks)

✓ [1] for $^{20}C_{13}$ or $^{25}C_{13}$

✓ [1] for final answer

$$\frac{^{20}C_{13}}{^{25}C_{13}} = \frac{12}{805}$$

(g) (3 marks)

✓ [1] for use of binomial theorem

✓ [1] for value of k

✓ [1] for the correct term

$$\begin{aligned}\left(3x-\frac{1}{2x^2}\right)^9 &= \sum_{k=0}^{9}\binom{9}{k}(3x)^{9-k}\left(-\frac{1}{2x^2}\right)^k\\ &= \sum_{k=0}^{9}\binom{9}{k}3^{9-k}\left(-\frac{1}{2}\right)^k x^{9-k}\times x^{-2k}\\ &= \sum_{k=0}^{9}\binom{9}{k}3^{9-k}\left(-\frac{1}{2}\right)^k x^{9-3k}\end{aligned}$$

The term independent of $x$is when $k=3$.

$$\therefore \binom{9}{3}3^6\left(-\frac{1}{2}\right)^3 = -7654.5 \text{ or } -\frac{15309}{2}$$

Question 12

(a) (2 marks)

✓ [1] for $P'(x)$

✓ [1] for showing α is a zero of $P'(x)$ by substitution

$$\begin{aligned}P(x) &= (x-\alpha)^2Q(x)\\ P'(x) &= 2(x-\alpha)Q(x)+(x-\alpha)^2Q'(x)\\ P'(\alpha) &= 0+0\\ &= 0\end{aligned}$$

$$\therefore x=\alpha \text{is a zero of } P'(x).$$

(b) i. (2 marks)

✓ [1] for $\overrightarrow{AB}$

✓ [1] for final answer

$$\overrightarrow{AB} = \begin{pmatrix}3\\-2\end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{9+4} = \sqrt{13}$$

$$\therefore \text{unit vector is } \frac{1}{\sqrt{13}}\begin{pmatrix}3\\-2\end{pmatrix}$$

ii. (2 marks)

✓ [1] for use of projection formula

✓ [1] for final answer

Applying the projection of $\underset{\sim}{F}$ on to $\overrightarrow{AB}$, such that

$$\begin{aligned}\text{proj}_{\underset{\sim}{F}}\,\overrightarrow{AB} &= \frac{\underset{\sim}{F}\cdot\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|^2}\overrightarrow{AB}\\ &= \frac{\begin{pmatrix}2\\5\end{pmatrix}\cdot\begin{pmatrix}3\\-2\end{pmatrix}}{13}\begin{pmatrix}3\\-2\end{pmatrix}\\ &= \frac{6-10}{13}\begin{pmatrix}3\\-2\end{pmatrix}\\ &= -\frac{4}{13}\begin{pmatrix}3\\-2\end{pmatrix}\end{aligned}$$

(c) (3 marks)

✓ [1] for A^2 in terms of $\cos^2\theta$

✓ [1] for substituting $\cos^2\theta = \dfrac{(\underset{\sim}{a}\cdot\underset{\sim}{b})^2}{|\underset{\sim}{a}|^2\,|\underset{\sim}{b}|^2}$ into A^2

✓ [1] for final result

$$
\begin{aligned}
A &= \frac{1}{2}\left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\sin\theta \\
A^2 &= \frac{1}{4}\left|\underset{\sim}{a}\right|^2\left|\underset{\sim}{b}\right|^2\sin^2\theta \\
&= \frac{1}{4}\left|\underset{\sim}{a}\right|^2\left|\underset{\sim}{b}\right|^2(1-\cos^2\theta) \\
\because \cos\theta &= \frac{(\underset{\sim}{a}\cdot\underset{\sim}{b})}{\left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|}, \\
\cos^2\theta &= \frac{(\underset{\sim}{a}\cdot\underset{\sim}{b})^2}{\left|\underset{\sim}{a}\right|^2\left|\underset{\sim}{b}\right|^2} \\
A^2 &= \frac{1}{4}\left|\underset{\sim}{a}\right|^2\left|\underset{\sim}{b}\right|^2\left(1-\frac{(\underset{\sim}{a}\cdot\underset{\sim}{b})^2}{\left|\underset{\sim}{a}\right|^2\left|\underset{\sim}{b}\right|^2}\right) \\
&= \frac{1}{4}\left(\left|\underset{\sim}{a}\right|^2\left|\underset{\sim}{b}\right|^2-(\underset{\sim}{a}\cdot\underset{\sim}{b})^2\right) \\
&= \frac{1}{4}\left((\underset{\sim}{a}\cdot\underset{\sim}{a})(\underset{\sim}{b}\cdot\underset{\sim}{b})-(\underset{\sim}{a}\cdot\underset{\sim}{b})^2\right) \\
\therefore A &= \frac{1}{2}\sqrt{(\underset{\sim}{a}\cdot\underset{\sim}{a})(\underset{\sim}{b}\cdot\underset{\sim}{b})-(\underset{\sim}{a}\cdot\underset{\sim}{b})^2}
\end{aligned}
$$

(d) i. (2 marks)

- ✓ [1] for $\underset{\sim}{v}$
- ✓ [1] for final answer

$$\underset{\sim}{v} = \begin{pmatrix} d\frac{7}{\sqrt{2}} \\ d\frac{7}{\sqrt{2}} - 10t \end{pmatrix}$$

$$\text{When } \frac{7}{\sqrt{2}} - 10t = 0,$$

$$t = \frac{7}{10\sqrt{2}} = \frac{7\sqrt{2}}{20}$$

ii. (2 marks)

- ✓ [1] for maximum height from the initial position
- ✓ [1] for final answer

$$y = \frac{7}{\sqrt{2}} \times \frac{7\sqrt{2}}{20} - 5\left(\frac{7\sqrt{2}}{20}\right)^2 = 1.225$$

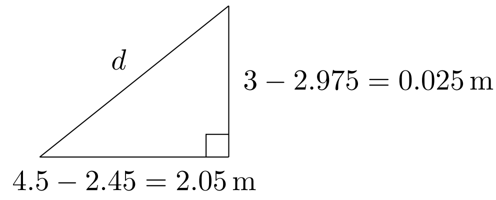
$$\therefore 1.225 + 1.75 \approx 2.98\,\text{m}$$

iii. (2 marks)

- ✓ [1] for horizontal distance from the initial position
- ✓ [1] for final answer

$$x = \frac{7}{\sqrt{2}} \times \frac{7\sqrt{2}}{20} = \frac{49}{20} = 2.45\,\text{m}$$

Let d be the distance between the centre of the ball and the centre of the ring.



$$d = \sqrt{2.05^2 + 0.025^2} = 2.05\,\text{m}$$

Question 13

(a) (3 marks)

- ✓ [1] for separation of variables and integration
- ✓ [1] for finding the value of constant
- ✓ [1] for simplifying the equation

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{2y}{x} \\
\frac{1}{2}\int\frac{1}{y}dy &= \int\frac{1}{x}dx \\
\frac{1}{2}\ln|y| &= \ln|x| + c
\end{aligned}
$$

$$\because \text{When } x = 1, y = -1,$$

$$c = 0$$

$$
\begin{aligned}
\therefore \frac{1}{2}\ln(-y) &= \ln x \\
\ln(-y) &= \ln x^2 \\
-y &= x^2 \\
y &= -x^2
\end{aligned}
$$

(b) (3 marks)

- ✓ [1] for proving the base case
- ✓ [1] for use of the assumption
- ✓ [1] for final result

- Prove true for $n = 1$:

  $$3^3 + 2^3 = 27 + 8 = 35$$

  $\therefore$ true for $n = 1$.
- Assume true for $n = k$

  $$3^{3k} + 2^{k+2} = 5M,\ \text{where } M \text{ is a positive integer}$$

- Prove true for $n = k+1$

  RTP: $3^{3k+3} + 2^{k+3}$ is divisible by 5.

$$\begin{aligned} & 3^{3k+3} + 2^{k+3} \\ &= 3^{3k} \times 3^3 + 2^{k+3} \\ &= (5M - 2^{k+2}) \times 27 + 2^{k+2} \times 2 \\ &\text{... from the assumption} \\ &= 135M - 27 \times 2^{k+2} + 2 \times 2^{k+2} \\ &= 135M - 25 \times 2^{k+2} \\ &= 5(27M - 5 \times 2^{k+2}) \\ &= 5N,\ \text{where } N \text{ is a positive integer} \end{aligned}$$

  $\therefore$ true for $n = k+1$.

- Conclusion
  By mathematical induction, the statement is true for all positive integers.

(c) i. (1 mark)

$$\begin{aligned} \frac{dT}{dt} &= -k \times (I - A)e^{-kt} \\ &= -k(T - A) \end{aligned}$$

$\therefore T = A + (I - A)e^{-kt}$ satisfies the differential equation.

ii. (2 marks)

✓ [1] for value of k

✓ [1] for final answer

$$\begin{aligned} I &= 1500, A = 20 \\ T &= 20 + 1480e^{-kt} \end{aligned}$$

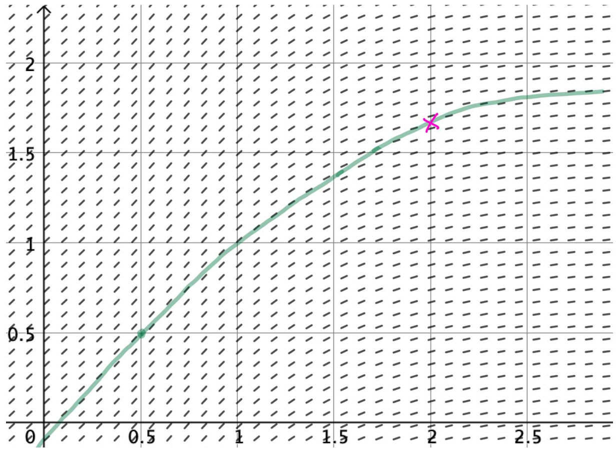
When $t = 5, T = 1200$

$$\begin{aligned} 1200 &= 20 + 1480e^{-5k} \\ \frac{118}{148} &= e^{-5k} \\ \therefore k &= -\frac{1}{5} \ln \frac{59}{74} \end{aligned}$$

When $t = 60$,

$$\begin{aligned} T &= 20 + 1480e^{-60k}, \qquad \text{where } k = -\frac{1}{5}\ln\frac{59}{74} \\ &= 118°C \text{ (3 sig. fig).} \end{aligned}$$

(d) i. (1 mark)



ii. (1 mark)

✓ [1] for a value in $1.6 \leq y \leq 1.8$

$$y \approx 1.7$$

(e) (3 marks)

✓ [1] for volume when the area under the curve is rotated

✓ [1] for volume of cone

✓ [1] for final answer

Let V_1 be the volume of the solid when the area under $y = \frac{4\sqrt{2}}{\pi}x$ is rotated and V_2 be the volume of the cone.

$$\begin{aligned} V_1 &= \pi \int_0^{\frac{\pi}{4}} \sec^2 x \, dx \\ &= \pi \left[\tan x\right]_0^{\frac{\pi}{4}} \\ &= \pi \\ V_2 &= \frac{1}{3} \times 2\pi \times \frac{\pi}{4} \\ &= \frac{\pi^2}{6} \\ \therefore V &= V_1 - V_2 \\ &= \pi - \frac{\pi^2}{6} \end{aligned}$$

OR alternatively,

$$\begin{aligned}
V &= \pi \int_0^{\frac{\pi}{4}} \left(\sec^2 x - \frac{32}{\pi^2} x^2 \right)\, dx \\
&= \pi \left[\tan x - \frac{32x^3}{3\pi^2} \right]_0^{\frac{\pi}{4}} \\
&= \pi \left(1 - \frac{32}{3\pi^2} \times \frac{\pi^3}{64} \right) \\
&= \pi \left(1 - \frac{\pi}{6} \right) \\
&= \pi - \frac{\pi^2}{6}
\end{aligned}$$

Question 14

(a) (4 marks)

- ✓ [1] for changing limits
- ✓ [1] for expressing the integral in terms of u
- ✓ [1] for integrating correctly
- ✓ [1] for final answer

$$\int_1^9 \frac{\sqrt{x}+2}{\sqrt{x}\sqrt{1+\sqrt{x}}} dx$$

$$u = 1 + \sqrt{x} \quad \sqrt{x} = u - 1$$

$$du = \frac{1}{2\sqrt{x}} dx \quad dx = 2\sqrt{x} du$$

When $x = 9, u = 4$
When $x = 1, u = 2$

$$\begin{aligned}
&= \int_2^4 \frac{u+1}{(u-1)\sqrt{u}} \times 2(u-1) du \\
&= 2 \int_2^4 u^{\frac{1}{2}} + u^{-\frac{1}{2}} du \\
&= 2 \left[\frac{2}{3} u^{\frac{3}{2}} + 2u^{\frac{1}{2}} \right]_2^4 \\
&= 2 \left(\frac{2}{3} \times 8 + 2 \times 2 - \frac{2}{3} \times 2\sqrt{2} - 2\sqrt{2} \right) \\
&= 2 \left(\frac{28}{3} - \frac{10}{3}\sqrt{2} \right) \\
&= \frac{56}{3} - \frac{20}{3}\sqrt{2}
\end{aligned}$$

(b) i. (2 marks)

- ✓ [1] for domain
- ✓ [1] for range

Domain:

$$-1 \le h - \frac{\sqrt{3}}{2} \le 1$$

$$-1 + \frac{\sqrt{3}}{2} \le h \le 1 + \frac{\sqrt{3}}{2}$$

However, since $h \ge 0$,

$$0 \le h \le 1 + \frac{\sqrt{3}}{2}$$

Range:

$$\because 0 \le \cos^{-1}\left(h - \frac{\sqrt{3}}{2} \right) \le \pi$$

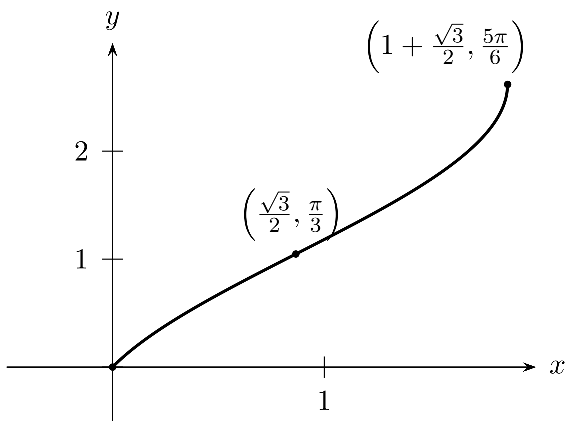
$$-\frac{\pi}{6} \le \frac{5\pi}{6} - \cos^{-1}\left(h - \frac{\sqrt{3}}{2} \right) \le \frac{5\pi}{6}$$

However, since $V \ge 0$

$$0 \le \frac{5\pi}{6} - \cos^{-1}\left(h - \frac{\sqrt{3}}{2} \right) \le \frac{5\pi}{6}$$

ii. (3 marks)

- ✓ [1] for shape
- ✓ [1] for endpoints
- ✓ [1] for point of inflexion



iii. (1 mark)

$$\because 50\,\text{L} = \frac{1}{20}\ \text{m}^3,$$

$$V = \frac{1}{20}t \text{ or } V = 0.05t$$

iv. (1 mark) When $V = \frac{5\pi}{6}$,

$$\begin{aligned}
\frac{5\pi}{6} &= \frac{1}{20}t \\
\therefore t &= \frac{5\pi}{6} \times 20 \\
t &= \frac{50\pi}{3}
\end{aligned}$$

v. (3 marks)

- ✓ [1] for $\frac{dV}{dh}$
- ✓ [1] for use of chain rule
- ✓ [1] for final answer

$$\frac{dV}{dh} = \frac{1}{\sqrt{1 - \left(h - \frac{\sqrt{3}}{2}\right)^2}}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \sqrt{1 - \left(h - \frac{\sqrt{3}}{2}\right)^2} \times \frac{1}{20}$$

When $h = \sqrt{3}$,

$$\frac{dh}{dt} = \sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2} \times \frac{1}{20} \quad = \frac{1}{2} \times \frac{1}{20}$$

$$= \frac{1}{40}$$

$\therefore \frac{1}{40}$ or 0.025 m/min.